ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 1

MATH00030

TRIMESTER 1 2023/2024

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Contents

6. Differential Calculus

6.1. **Differentiation from First Principles**.

In this chapter, we will introduce the concept of differentiation and indicate what it does. This chapter and the next one on integral calculus are introductory chapters and we will build on them in the second trimester, so it is important to be comfortable with the material before then.

We will start with the formal definition and then explain what it means.

Definition 6.1.1 (Derivative)**.** Let $f\colon (a,b) \to \mathbb{R}$, then the *derivative* of f at $x \in (a,b)$ is defined to be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \tag{1}$$

if this limit exists.
When we find the derivative of a function, we say we *differentiate* it. The process is called *differentiation*.

Remark 6.1.2. In the definition there are the words 'if this limit exists'. If the limit doesn't exist (we won't look at how this can happen here) then the derivative of f does not exist at x. There are many functions that are not differentiable but we won't study these in this course.

Let us now examine what the definition means. If we look at Figure 1, we will see that the slope of the line (See Chapter 2) from the point $(x, f(x))$ to the point $(x+h, f(x+h))$ is just $\dfrac{f(x+h)-f(x)}{h}$.

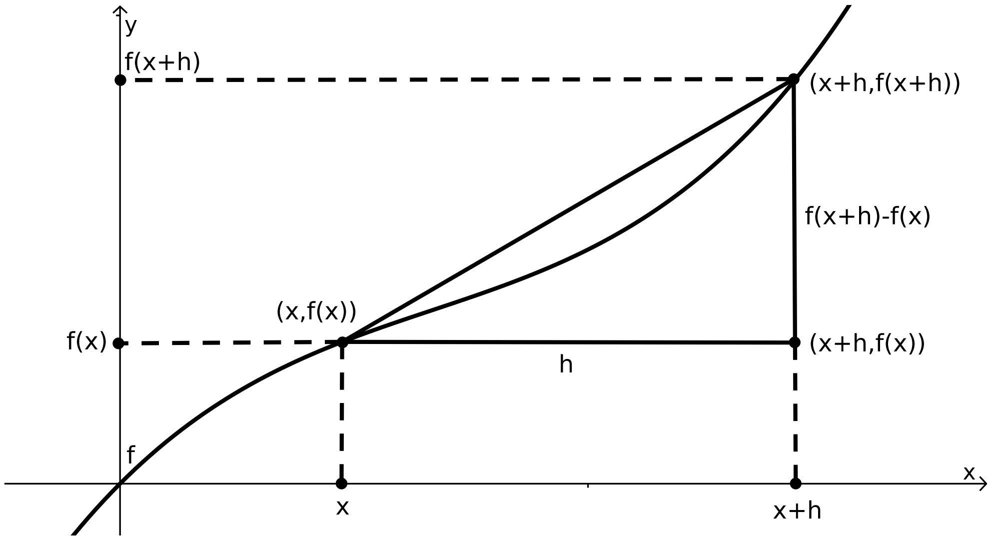

FIGURE 1. Slope of a line connecting $(x, f(x))$ to $(x+h, f(x+h))$.

When we differentiate a function at a point x, what we are really doing is to make h smaller and smaller in $\dfrac{f(x+h)-f(x)}{h}$ and see what happens to it (this is what the $\lim\limits_{h\to 0}$ is telling us to do). What this means graphically is shown in Figure 2. Hopefully this figure will convince you that the derivative of f at x is the slope of the tangent line to the function f at the point x. This also tells us when a function is not differentiable. At any point where the tangent line does not exist (for example where the graph has a kink in it) the derivative doesn't exist either.

Dr. John Sheekey has prepared an interactive GeoGebra worksheet which shows the process of finding a derivative. It can be found at
http://www.ucd.ie/msc/access/differentiationfromfirstprinciples/

Remark 6.1.3. Often the derivative of a function f will be denoted by $\dfrac{dy}{dx}$, $\dfrac{df}{dx}$, $\dfrac{d}{dx}(f)$ or even f_x rather than $f'(x)$. Also note that sometimes completely different letters may be used, so you may see things like $g'(x)$ or $\dfrac{dg}{dx}$ or indeed $\dfrac{dx}{dy}$, where the roles of x and y have been reversed. All these different notations mean exactly the same thing. They only exist since calculus was developed by different mathematicians and the various notations have persisted.

While all this may seem like a lot of effort to go to just to find the gradient of a tangent to a curve, it is extremely important since it arises in numerous different

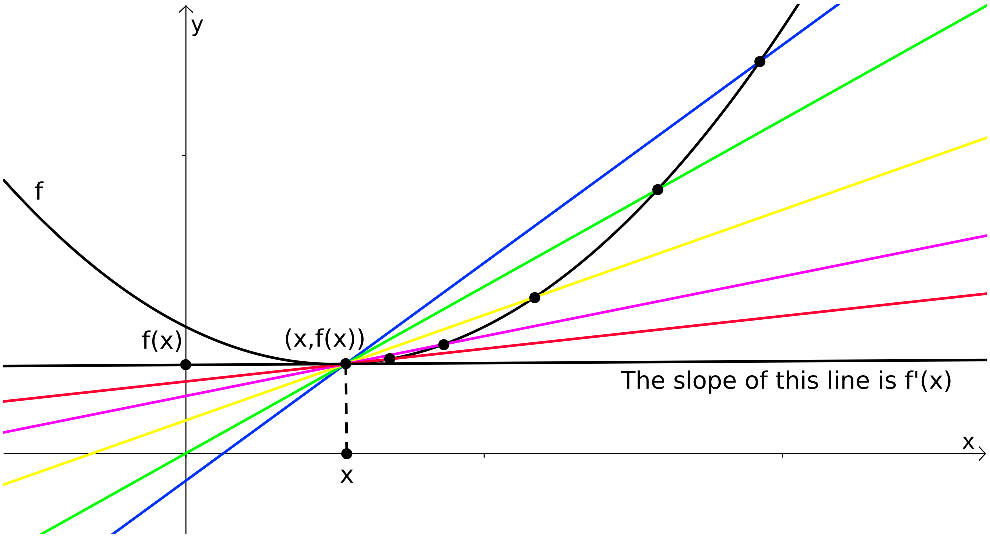

FIGURE 2. Geometric meaning of the derivative.

areas. Whenever you want to find the rate of change of something then calculus will come in handy. For example, if you have a function representing the position of an object, then the derivative will represent the velocity of the object. Similarly if you have a function representing the velocity of an object then the derivative of this function will represent the acceleration of the object.
Now let us differentiate some functions from 'first principles'. This means that we are going to use (1) to perform the differentiation, rather than a table of derivatives.

Example 6.1.4.

(1) Find the derivative of the function $f(x) = 3x$.
Using (1), we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h) - 3x}{h} \\
&= \lim_{h\to 0} \frac{3x + 3h - 3x}{h} \\
&= \lim_{h\to 0} \frac{3h}{h} \\
&= \lim_{h\to 0} 3 \\
&= 3.
\end{aligned}
$$

Thus $f'(x) = 3$.

(2) Find the derivative of the function $f(x) = x^2$.
Using (1), we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x. \end{aligned}$$

Thus $f'(x) = 2x$.

(3) Find the derivative of the function $f(x) = 2x^3$.
Using (1), we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2(x+h)^3 - 2x^3}{h} \\ &= \lim_{h\to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h} \\ &= \lim_{h\to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x^3}{h} \\ &= \lim_{h\to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} \\ &= \lim_{h\to 0} 6x^2 + 6xh + 2h^2 \\ &= 6x^2. \end{aligned}$$

Thus $f'(x) = 6x^2$.

(4) Find the derivative of the function $f(x) = 3x^2 - 2x + 1$.
Using (1), we have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{3(x+h)^2 - 2(x+h) + 1 - (3x^2 - 2x + 1)}{h} \\
&= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 2(x+h) + 1 - (3x^2 - 2x + 1)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h + 1 - 3x^2 + 2x - 1}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h} \\
&= \lim_{h \to 0} 6x + 3h - 2 \\
&= 6x - 2.
\end{aligned}$$

Thus $f'(x) = 6x - 2$.

(5) Find the derivative of the function $f(x) = c$, where c is a constant.
Using (1), we have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{c - c}{h} \\
&= \lim_{h \to 0} 0 \\
&= 0.
\end{aligned}$$

Thus $f'(x) = 0$.

6.2. Some Common Derivatives.

From now on, we will concentrate on the actual mechanics of differentiation, rather than worrying about differentiating functions from first principles. In Table 1 there is a list of derivatives that you should be able to use. Note that a formula sheet will be provided in the exam, so you should concentrate on learning how to use them, not on memorising them. In the table x represents a variable while a represents a constant.

Warning 6.2.1.

(1) Note that the derivative of $\ln(ax)$ is $\dfrac{1}{x}$, no matter what the value of a is (provided $ax > 0$). This is **NOT** a typo.
(2) We need $ax > 0$ for the derivative of $\ln(ax)$ to ensure $\ln(ax)$ exists.
(3) Also note that the derivatives of $\sin(ax)$ and $\cos(ax)$ are only valid if x is in radians. If x is in degrees then extra constants would be needed but in practice we **NEVER** use degrees when differentiating.

$f(x)$	$f'(x)$	Comments
c	0	Here c is any real number
x^n	nx^{n-1}	
e^{ax}	ae^{ax}	
$\ln(ax)$	$\dfrac{1}{x}$	Here we must have $ax > 0$
$\sin(ax)$	$a\cos(ax)$	
$\cos(ax)$	$-a\sin(ax)$	Note the change of sign

TABLE 1. Some common derivatives

As usual, a few examples will make things clearer. Please see Table 2.

6.3. The Sum and Multiple Rules.

Although the list of derivatives in Table 1 is very useful, we would not get very far if these were the only functions we could differentiate. Luckily there are rules that allow us to differentiate more complicated functions. The first of these allows us to differentiate sums of functions.

Theorem 6.3.1 (The Sum Rule for Differentiation)**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$, then the derivative of $f+g$ at $x \in (a,b)$ is given by*

$$(f+g)'(x) = f'(x) + g'(x),$$

provided these derivatives exist.

All this says is that if we want to differentiate a sum of two functions then all we have to do is differentiate them separately and add the derivatives.

Here are a couple of examples of the use of the Sum Rule.

Example 6.3.2.

(1) Find the derivative of $f(x) = x^2 + \sin(2x)$.

$$f'(x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin(2x)) = 2x + 2\cos(2x).$$

(2) Find the derivative of $f(x) = \ln(2x) + e^{-3x}$.
Provided $x > 0$ (so that the derivative of the first term exists),

$$f'(x) = \frac{d}{dx}(\ln(2x)) + \frac{d}{dx}(e^{-3x}) = \frac{1}{x} - 3e^{-3x}.$$

The second rule that will enable us to differentiate a larger range of functions is the Multiple Rule.

Theorem 6.3.3 (The Multiple Rule for Differentiation)**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $c \in \mathbb{R}$, then the derivative of cf at $x \in (a,b)$ is given by*

$$(cf)'(x) = cf'(x),$$

provided the derivative of f exists.

$f(x)$	$f'(x)$	Comments
0	0	Note the derivative of 0 is 0
2	0	
-4	0	
$-\pi$	0	$-\pi$ is just a number
e	0	e is just a number
$\cos(1)$	0	$\cos(1)$ is just a number
x	1	Since $x = x^1$, $n = 1$ giving $1x^0 = 1$
x^3	$3x^2$	Here we take $n = 3$
x^{-4}	$-4x^{-5} = -\dfrac{4}{x^5}$	Here we take $n = -4$
x^{π}	$\pi x^{\pi-1}$	π is just a number
x^{-e}	$-ex^{-e-1} = -\dfrac{e}{x^{e+1}}$	e is just a number
e^x	e^x	Here we take $a = 1$
e^{5x}	$5e^{5x}$	Here we take $a = 5$
e^{-7x}	$-7e^{-7x}$	Here we take $a = -7$
e^{ex}	$e \cdot e^{ex} = e^{ex+1}$	Here we take $a = e$
$\ln(x)$	$\dfrac{1}{x}$	Here we must have $x > 0$
$\ln(5x)$	$\dfrac{1}{x}$	Here we must have $x > 0$
$\ln(-5x)$	$\dfrac{1}{x}$	Here we must have $x < 0$
$\sin(x)$	$\cos(x)$	Here we take $a = 1$
$\sin(3x)$	$3\cos(3x)$	Here we take $a = 3$
$\sin(-2x)$	$-2\cos(-2x)$	Here we take $a = -2$
$\sin(-\pi x)$	$-\pi\cos(-\pi x)$	Here we take $a = -\pi$
$\cos(x)$	$-\sin(x)$	Here we take $a = 1$
$\cos(4x)$	$-4\sin(4x)$	Here we take $a = 4$
$\cos(-5x)$	$5\sin(-5x)$	Note $-(-5) = +5$
$\cos(\pi x)$	$-\pi\sin(\pi x)$	Here we take $a = \pi$

TABLE 2. Some examples of derivatives

All this says is that if we want to differentiate a constant multiple of a function, then all we have to do is first differentiate the function and then multiply by the constant.

Here are a couple of examples of the Multiple Rule.

Example 6.3.4.

(1) Find the derivative of $f(x) = 5x^3$.

$$f'(x) = 5 \times \frac{d}{dx}(x^3) = 5 \times 3x^2 = 15x^2.$$

(2) Find the derivative of $f(x) = -3\cos(2x)$.

$$f'(x) = -3 \times \frac{d}{dx}(\cos(2x)) = -3 \times (-2\sin(2x)) = 6\sin(2x).$$

Warning 6.3.5. The Multiple Rule can only be used to differentiate a product of a number and a function. If we want to differentiate the product of two functions, then we have to use the Product Rule which we will study in the second trimester.

Of course we are free to use both the Sum and Multiple Rules to differentiate a function and the following are a couple of examples of this.

Example 6.3.6.

(1) Find the derivative of $f(x) = 5x^2 - 4x + 3$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(5x^2) + \frac{d}{dx}(-4x) + \frac{d}{dx}(3) \quad \text{(using the Sum Rule)} \\ &= 5\frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) + \frac{d}{dx}(3) \quad \text{(using the Multiple Rule)} \\ &= 5(2x) - 4(1) + 0 \\ &= 10x - 4. \end{aligned}$$

(2) Find the derivative of $f(x) = -e^{-2x} - 2\cos(-3x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(-e^{-2x}) + \frac{d}{dx}(-2\cos(-3x)) \quad \text{(using the Sum Rule)} \\ &= -\frac{d}{dx}(e^{-2x}) + (-2)\frac{d}{dx}(\cos(-3x)) \quad \text{(using the Multiple Rule)} \\ &= -(-2e^{-2x}) - 2(-3(-\sin(-3x))) \\ &= 2e^{-2x} - 6\sin(-3x). \end{aligned}$$